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Diffusive Traversal Time of a One-Dimensional Medium

C. W. Gardiner[1]

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It is shown that the mean time to traverse a medium in a range a, b from b to a is given by $T=(a-b)^2/3D$, where D is the diffusion coefficient.

KEY WORDS: Fokker–Planck equation; diffusive; traversal; diffusive traversal; mean exit time.

In a recent paper, Landauer and Büttiker[(1)] considered the problem of a particle diffusing from one side of a medium to the other as part of their investigation of magnetically induced interference in a metallic wire. They derived a quite simple formula for a particle to diffuse from point a to point b under the condition that it does not leave the interval between a and b until its exit at b.

In fact, this is the limitation case of the general problem: if a diffusing particle is located initially at x within a medium that extends from a to b, what is the mean time to exit through the end a?

This problem is treated in Section 5.2.8 of Ref. 2, and in summary, the procedure is as follows. Let

$$\pi_a(x) = \text{probability of exit through } a \tag{1}$$

$$\pi_b(x) = \text{probability of exit through } b \tag{2}$$

$$T(a, x) = \text{mean time to exit through } a \tag{3}$$

Then it is shown in this reference that $\pi_a(x)$ and $T(a, x)$ are given by solving

$$\tfrac{1}{2}D\,\partial_x^2[\pi_a(x)\,T(a, x)] = -\pi_a(x) \tag{4}$$

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[1] Physics Department, University of Waikato, Hamilton, New Zealand.

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subject to the boundary conditions

$$\pi_a(a)\, T(a, a) = \pi_a(b)\, T(a, b) = 0 \tag{5}$$

and

$$\tfrac{1}{2}D\, \partial_x^2[\pi_a(x)] = 0 \tag{6}$$

subject to the boundary conditions

$$\pi_a(a) = 1, \qquad \pi_a(b) = 0 \tag{7}$$

The equations are straightforwardly integrated to give

$$\pi_a(x) = \frac{b-x}{b-a} \tag{8}$$

and

$$\pi_a(x)\, T(a, x) = \frac{(x-b)(x+a-2b)(x-a)}{3D(b-a)} \tag{9}$$

and hence

$$T(a, x) = (x-a)(2b-x-a)/3D \tag{10}$$

In the limiting case of $x \to b$ the probability of exit through a is zero, as shown by (7); this is obvious, since a particle placed on the edge of the medium is certain to exit there immediately. Nevertheless, in the limit that x is very close to b, the mean time to make the exit at a is well defined:

$$T(a, b) \to (a-b)^2/3D \tag{11}$$

If the diffusing medium is an approximation to a one-dimensional random walk, then

$$b - a = Ll \tag{12}$$

$$D = l^2/\tau \tag{13}$$

where l is the distance between lattice points, L is the number of lattice points traversed, and τ is the time between steps. Then

$$T(a, b) = \tfrac{1}{3}L^2\tau \tag{14}$$

and hence the mean number of steps is $L^2/3$, as was derived by Landauer and Büttiker.[1]

REFERENCES

1. R. Landauer and M. Büttiker, *Phys. Rev. B*, to be published.
2. C. W. Gardiner, *A Handbook of Stochastic Methods* (Springer, Heidelberg, 1986).

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